(Defn.) Analytic, Holomorphic and Regular Function:

A single valued function $f(z)$ is said to be analytic in a domain D is said to be analytic or holomorphic in D.

If a function $f(z)$ is analytic at some point in every n'hood of point z_0, except at the point z_0 then the point z_0 is known as an isolated singularity. A function $f(z)$ is said to have removable singularity at a point z_0 of the range of definition of f. If the function $f(z)$ is not differentiable at z_0 but can be made analytic by nearly a single a sutable value of the function at z_0.

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Sufficient Condition for $f(z)$ to be analytic.

Q.No → State and Prove of Sufficient Condition for $f(z)$ to be analytic.

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Or Q.No → Obtain a set of sufficient Condition for the function $f(z) = u + iv$ to be analytic in a given domain.

Ans. Statement:- The Continuous one-valued function $f(z)$ is analytic in a domain D if the four Partial derivatives u_x, v_x, u_y, v_y exist, are Continuous and Satisfy the Cauchy-Riemann equations at each Point of D.

Proof:- Since, $u = u(x, y)$, we have

$$u + \Delta u = u(x + \Delta x, y + \Delta y).$$

$$\therefore \Delta u = u(x + \Delta x, y + \Delta y) - u(x, y)$$

$$= \{u(x + \Delta x, y + \Delta y) - u(x + \Delta x, y)\} + u(x + \Delta x, y) - u(x, y)$$

$$= \Delta y \, u_y(x + \Delta x, y + \theta \Delta y) + \Delta x \, u_x(x + \theta' \Delta x, y) \quad \text{--- (1)}$$

where; $0 < \theta < 1$ & $0 < \theta' < 1$.

By the mean value theorem.

Since, u_x & u_y are Continuous in the given region for a given $\epsilon > 0$, $\exists \, \delta > 0$, such that

$$|u_y(x + \Delta x, y + \theta \Delta y) - u_y(x, y)| < \epsilon$$

and, $|u_x(x + \theta' \Delta x, y) - u_x(x, y)| < \epsilon$

For $|\Delta x| < \delta$, $|\Delta y| < \delta$.

Let $u_y(x + \Delta x, y + \theta \Delta y) - u_y(x, y) = \alpha$

and $u_x(x + \theta' \Delta x, y) - u_x(x, y) = \beta$.

Then, we have from (1),

$$\Delta u = [u_y(x,y)+\alpha]\Delta y + [u_x(x,y)+\beta]\Delta x$$

Similarly, it can be shown that,

$$\Delta v = [v_y(x,y)+\alpha']\Delta y + [v_x(x,y)+\beta']\Delta x$$

where $|\alpha'| < \epsilon'$ and $|\beta'| < \epsilon'$.

Let the given function $w = f(z) = u + iv$

$\therefore w + \Delta w = f(z+\Delta z) = (u+\Delta u) + i(v+\Delta v)$

$\therefore \Delta w = f(z+\Delta z) - f(z) = \Delta u + i\Delta v$.

Hence, $\dfrac{\Delta w}{\Delta z} = \dfrac{f(z+\Delta z)-f(z)}{\Delta z} = \dfrac{\Delta u + i\Delta v}{\Delta x + i\Delta y}$

$$= \frac{[u_y(x,y)+\alpha]\Delta y + [u_x(x,y)+\beta]\Delta x + i[v_y(x,y)+\alpha']\Delta y + i[v_x(x,y)+\beta']\Delta x}{\Delta x + i\Delta y}$$

$$\cancel{= \frac{[u_y\Delta y + u_x\Delta x + \alpha\Delta y + \beta\Delta x] + i(v_y\Delta y + v_x\Delta x + \alpha'\Delta y + \beta'\Delta x)}{\Delta x + i\Delta y}}$$

$$= \cancel{-v_x\Delta y + u_x\Delta x + \cdots}$$

$$= \frac{[-v_x(x,y)+\alpha]\Delta y + [u_x(x,y)+\beta]\Delta x + i[u_x(x,y)+\alpha']\Delta y + i[v_x(x,y)+\beta']\Delta x}{\Delta x + i\Delta y} \quad [\because u_x = v_y \ \& \ u_y = -v_x]$$

$$= \frac{u_x + iv_x + \alpha\Delta y + \beta\Delta x + i\alpha'\Delta y + i\beta'\Delta x}{\Delta x + i\Delta y}$$

$$\therefore \left|\frac{\Delta w}{\Delta z} - (u_x + iv_x)\right| = \left|\frac{(\alpha+i\alpha')\Delta y + (\beta+i\beta')\Delta x}{\Delta x + i\Delta y}\right|$$

$$\leq \frac{|\alpha+i\alpha'||\Delta y| + |\beta+i\beta'||\Delta x|}{|\Delta x + i\Delta y|}$$

$$\leq |\alpha+i\alpha'| + |\beta+i\beta'| \quad \because |\Delta y| \leq |\Delta x + i\Delta y| \ \& \ |\Delta x| \leq |\Delta x + i\Delta y|$$

$$\le |\alpha| + |i\alpha'| + |\beta| + |i\beta'|$$

$$= |\alpha| + |\alpha'| + |\beta| + |\beta'| < \epsilon + \epsilon + \epsilon' + \epsilon' = 2(\epsilon + \epsilon')$$

$$\therefore \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = u_x + i v_x$$

Therefore, $f(z)$ is differentiable and $f'(z) = u_x + i v_x$.

QNo→ Prove that the reail and imaginary parts of an analytic function satisfy Laplace's differential Equation.

Proof:- We know that

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{—— (1)}$$ is the Laplace differential equation,

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ \& \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \ \& \ \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$$

Hence, u Satisfies (1)

Again, $$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} \ \& \ \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x}$$

$$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y \partial x} = 0$$

That is v Satisfy (1), Hence the real & imaginary Part of an analytic function Satisfy Laplace's differential equation.

Exam

Q.No→ If $w = f(z) = u + iv$ be an analytic function of $z = x + iy$, show that the curves $u = C_1$, $v = \text{Const.}$ represented on the z-plane intersect at right angles.

Soln. Since, $f(z) = u + iv$ is regular function of z, the functions u & v would satisfy Cauchy - Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Multiplying these, we get

$$\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y} = 0$$

Which is the Condition that the curves $u = \text{Constt.}$ & $v = \text{Constt.}$ intersect at right angles as shown above.

Hence if $f(z)$ is regular function of z, then the curves

$$u = R[f(z)] = \text{Constt.}$$

and, $v = I[f(z)] = \text{Constt.}$

from an orthogonal System i.e. they intersect at right angles.